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Reduced basis technique for nonlinear vibration analysis of composite panels

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A reduced basis technique and a computational procedure are presented for the nonlinear free vibrations of composite panels. The computational procedure can be conveniently divided into two distinct steps. The first step involves the generation of various-order perturbation vectors using Linstedt–Poincaré perturbation technique. The second step consists of using the perturbation vectors as basis vectors, computing the amplitudes of these vectors, and the nonlinear frequency of vibration, via a direct variational procedure. The analytical formulation is based on a form of the geometrically nonlinear shallow shell theory with the effects of transverse shear deformation, rotatory inertia and anisotropic material behavior included. The panel is discretized by using mixed finite element models with the fundamental unknowns consisting of both the nodal displacements and the stress-resultant parameters of the panel. The potential of the proposed technique is discussed and its effectiveness is demonstrated by means of numerical examples.

1. Introduction

The physical understanding and the numerical simulation of the nonlinear vibrational response of laminated anisotropic plates has recently become the focus of intense efforts. This is because of the expanded use of composite construction in aerospace, automotive, shipbuilding and other industries and the need to establish practical limits of their dynamic load-carrying capabilities.

Experimental studies have been performed on laminated anisotropic panels by Mayberry and Bert [1]. Analytical, numerical and hybrid analytical–numerical techniques have been

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developed for studying the nonlinear vibrations of anisotropic and laminated plates. Review of some of the techniques are contained in survey papers [2–4] and in two monographs [5, 6]. In most of the analytical techniques reported in the literature, the spatial discretization is done by either Galerkin or Ritz methods, and the temporal integration is done by using either the method of harmonic balance or a perturbation technique [7–17]. Most of the numerical techniques reported are based on using finite elements for the spatial discretization, and the temporal integration by assuming harmonic variation of the response with time [18–24]. In [25, 26] the nonlinear vibration problem was approximated by a linearized vibration problem of a prestressed panel. The geometric stiffness matrix corresponds to an average configuration of the panel. In the hybrid numerical analytical techniques, either the spatial discretization is done by using Galerkin's method and the temporal integration performed by direct numerical integration technique; or the spatial discretization is done by finite elements and the method of harmonic balance is used, in conjunction with a perturbation technique, to determine the nonlinear frequency of vibration (see [27–31]). The mathematical models used in the cited references range from simplified Von Karman type plate models (with in-plane deformation, in-plane inertia and rotatory inertia terms neglected) to higher-order shear-deformation models. Although the aforementioned studies have contributed significantly to understanding the influence of nonlinearities on the vibrational response of anisotropic and laminated plates, they were mostly based on perturbing a single vibration mode, and did not account for the effect of coupling of the vibration modes on the nonlinear frequency. Only few reported studies have accounted for the coupling effect (see, for example, [32, 33]).

The present study focuses on the nonlinear free vibrations of multilayered composites, and accounts for the coupling between the vibration modes. Specifically, the objectives of the present paper are

(1) to present a reduced-basis technique and a computational procedure for the nonlinear free vibrations of composite panels; and
(2) to demonstrate the effectiveness of the proposed technique by means of numerical examples.

To sharpen the focus of the study, only simply-connected rectangular panels are considered in the numerical studies. However, the computational procedure is expected to be particularly effective for panels with complex geometry. The reduced-basis technique presented herein is an extension of the reduction techniques presented in [34–37] for solution of nonlinear static and thermal problems.

The analytical formulation is based on a form of the geometrically nonlinear shallow shell theory with the effects of transverse shear deformation, in-plane inertia, rotatory inertia and anisotropic material behavior included. The panel is discretized by using mixed finite element models with the fundamental unknowns consisting of both the nodal displacements and the stress resultant parameters of the panel.

2. Basic idea of the reduced-basis technique

2.1. Governing semi-discrete finite element equations

The nonlinear free vibrational response of the panel can be described by the following system of ordinary differential equations:

$$\begin{bmatrix} -[F] & [S] \\ [S]^{t} & 0 \end{bmatrix} \begin{Bmatrix} H \\ X \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [M] \end{bmatrix} \begin{Bmatrix} \ddot{H} \\ \ddot{X} \end{Bmatrix} + \begin{Bmatrix} G(X) \\ \bar{G}(H, X) \end{Bmatrix} = 0 , \tag{1}$$

where $\{H\}$ is the vector of stress-resultant parameters; $\{X\}$ is the vector of nodal displacements; $[F]$ is the global flexibility matrix; $[S]$ is the strain-displacement matrix; $\{G(X)\}$ and $\{\bar{G}(H, X)\}$ are vectors of nonlinear terms; superscript t denotes transposition; 0 refers to a zero submatrix; and a dot $(\cdot)$ refers to derivative with respect to the time t.

The application of the reduced basis technique to the solution of eq. (1) can be conveniently divided into two distinct steps: (1) generation of basis (or global approximation) vectors using the Linstedt–Poincaré perturbation technique; and (2) computation of the amplitudes of the basis vectors and the nonlinear vibration frequency via a direct variational technique. The procedure is described subsequently.

2.2. Generation of basis vectors

For the purpose of generating the basis vectors, a new independent variable $\tau = \omega t$ is introduced, where ω is the nonlinear circular frequency. The following expansion is used for $\Omega = \omega^2$, in terms of a small parameter ε:

$$\Omega(\varepsilon) = \sum_{i=0} \Omega^{(i)} \varepsilon^{i} . \tag{2}$$

Only the even values of i ($i = 0, 2, 4, \ldots$) are retained in the expansion. The vectors $\{H\}$ and $\{X\}$ are also expanded in perturbation series of the form

$$\begin{Bmatrix} H(t, \varepsilon) \\ X(t, \varepsilon) \end{Bmatrix} = \sum_{i=1} \begin{Bmatrix} H(\tau) \\ X(\tau) \end{Bmatrix}^{(i)} \varepsilon^{i} . \tag{3}$$

Each of the time-dependent perturbation vectors, $\{H(\tau)\}^{(i)}$ and $\{X(\tau)\}^{(i)}$, are expanded in a Fourier series in τ as follows:

$$\begin{Bmatrix} H(\tau) \\ X(\tau) \end{Bmatrix}^{(i)} = \sum_{m=0}^{i} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} \cos m\omega t . \tag{4}$$

The equations used in generating the basis vectors are obtained by substituting the expansions for $\{H\}$, $\{X\}$ and ω, eqs. (2)–(4), into (1); converting each term into the first power of cosine functions; and setting the coefficients of like terms of ε and m to zero. This leads to a recursive set of linear equations in $\{H\}^{(i,m)}$ and $\{X\}^{(i,m)}$ which, for individual elements, can be written in the following compact form:

$$\begin{bmatrix} -[F] & [S] \\ [S]^{t} & -m^2\omega^2[M] \end{bmatrix} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} = \begin{Bmatrix} R \\ Q \end{Bmatrix}^{(i,m)} . \tag{5}$$

The explicit forms of the right-hand side vectors $\{R\}^{(i,m)}$ and $\{Q\}^{(i,m)}$ for $1 \leq i \leq 4$, $0 \leq m \leq 4$ are given in Appendix A and Table 2. Note that the linear free vibration problem corresponds to $i = m = 1$. The nonzero vectors $\{H\}^{(i,m)}$ and $\{X\}^{(i,m)}$, associated with different combinations of i and m, are listed in Table 1. All of these vectors are associated with a single vibration mode (i.e., a prescribed pair of eigenvalue and eigenvector).

Table 1
Pairs of (i, m) for which the perturbation vectors are nonzero

$$\begin{Bmatrix} H \\ X \end{Bmatrix} = \left(\sum_{m=0}^{i} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} \cos m\omega t \right) \varepsilon^i$$

$i \backslash m$	0	1	2	3	4	5	6
1		1,1					
2	2,0		2,2				
3		3,1		3,3			
4	4,0		4,2		4,4		
5		5,1		5,3		5,5	
6	6,0		6,2		6,4		6,6

Table 2

i	m	$R_I^{(i,m)}$	$Q_{I'}^{(i,m)}$
2	0	$\frac{1}{2} N_{IJ'K'} X_{J'}^{(1,1)} X_{K'}^{(1,1)}$	$N_{KI'J'} H_K^{(1,1)} X_{J'}^{(1,1)}$
2	2		
3	1	$N_{IJ'K'}(2X_{J'}^{(1,1)} X_{K'}^{(2,0)} + X_{J'}^{(1,1)} X_{K'}^{(2,2)})$	$N_{KI'J'}(2H_K^{(1,1)} X_{J'}^{(2,0)} + 2H_K^{(2,0)} X_{J'}^{(1,1)} + H_K^{(1,1)} X_{J'}^{(2,2)}$ $+ H_K^{(2,2)} X_{J'}^{(1,1)}) + c_1 M_{I'J'} X_{J'}^{(1,1)}$
3	3	$N_{IJ'K'} X_{J'}^{(1,1)} X_{K'}^{(2,2)}$	$N_{KI'J'}(H_K^{(1,1)} X_{J'}^{(2,2)} + H_K^{(2,2)} X_{J'}^{(1,1)})$
4	0	$N_{IJ'K'}(X_{J'}^{(1,1)} X_{K'}^{(3,1)} + X_{J'}^{(2,0)} X_{K'}^{(2,0)}$ $+ \frac{1}{2} X_{J'}^{(2,2)} X_{K'}^{(2,2)})$	$N_{KI'J'}(H_K^{(1,1)} X_{J'}^{(3,1)} + H_K^{(3,1)} X_{J'}^{(1,1)} + 2H_K^{(2,0)} X_{J'}^{(2,0)} + H_K^{(2,2)} X_{J'}^{(2,2)})$
4	2	$N_{IJ'K'}(X_{J'}^{(1,1)} X_{K'}^{(3,1)} + X_{J'}^{(1,1)} X_{K'}^{(3,3)}$ $+ 2X_{J'}^{(2,0)} X_{K'}^{(2,2)})$	$N_{KI'J'}(H_K^{(1,1)} X_{J'}^{(3,1)} + H_K^{(3,1)} X_{J'}^{(1,1)} + H_K^{(1,1)} X_{J'}^{(3,3)}$ $+ H_K^{(3,3)} X_{J'}^{(1,1)} + 2H_K^{(2,0)} X_{J'}^{(2,2)} + 2H_K^{(2,2)} X_{J'}^{(2,0)}) + 4c_1 M_{I'J'} X_{J'}^{(2,2)}$
4	4	$N_{IJ'K'}(X_{J'}^{(1,1)} X_{K'}^{(3,3)} + \frac{1}{2} X_{J'}^{(2,2)} X_{K'}^{(2,2)})$	$N_{KI'J'}(H_K^{(1,1)} X_{J'}^{(3,3)} + H_K^{(3,3)} X_{J'}^{(1,1)} + H_K^{(2,2)} X_{J'}^{(2,2)})$

2.3. *Computation of amplitudes of basis vectors and nonlinear frequency*

The perturbation vectors $\{H(\tau)\}^{(i)}$ and $\{X(\tau)\}^{(i)}$ are now chosen as basis vectors and the vectors $\{H\}$ and $\{X\}$ are expressed as linear combinations of these vectors as follows:

$$\begin{Bmatrix} H \\ X \end{Bmatrix} = \sum_{i=1}^{r} \left(\sum_{m=0}^{i} \begin{Bmatrix} H \\ X \end{Bmatrix}^{(i,m)} \cos m\omega t \right) \psi_i , \tag{6}$$

where ψ_i are unknown parameters representing the amplitudes of the basis vectors, and r equals the total number of basis vectors used.

The Bubnov–Galerkin technique is then used, in conjunction with the method of harmonic balance, to approximate (1) by a reduced system of r nonlinear algebraic equations in ψ_i $(i = 1, \ldots, r)$ and ω. The additional equation needed to solve the system is obtained by prescribing either one of the displacement components (linear combination of ψ_i), or one of the parameters ψ_i. The form of the nonlinear algebraic equations in ψ_i and ω is given in Appendix B.

2.4. Comments on the selection of the basis vectors and the reduced basis technique

The following two comments concerning the selection of basis vectors and the reduced basis technique seem to be in order:

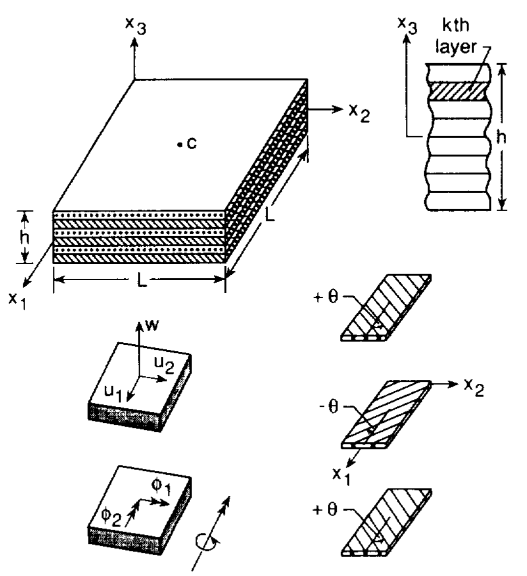
(1) The chosen set of basis vectors are linearly independent. Their generation using Linstedt–Poincaré perturbation technique requires the solution of a recursive set of linear algebraic equations.

(2) Whereas in the perturbation technique all the perturbation vectors are associated with a single mode, in the reduced-basis technique the basis vectors can be associated with more than one mode, thereby incorporating the effects of coupling between the different modes.

3. Numerical studies

To assess the effectiveness of the proposed reduced basis technique, a number of nonlinear vibration problems of multilayered composite panels have been solved using this technique. For each problem, the accuracy and convergence of the solutions obtained by the proposed technique were compared with those obtained by the perturbation technique, as well as with other numerical and approximate solutions, whenever available. Herein, the results are presented for typical two-layer cross-ply and angle-ply square panels (see Fig. 1).

The panels were discretized by using mixed finite element models with bicubic interpolation functions for each of the generalized displacements and stress resultants. The characteristics of the finite element model are given in [38]. Because of symmetry only one-quarter of the cross-ply panel, and one-half of the angle-ply panel, were analyzed and the appropriate symmetry/antisymmetry conditions were applied (see [39]). A 4×4 grid was used for the



	Material 1	Material 2
E_L/E_T	10.0	40.0
G_{LT}/E_T	0.333	0.5
G_{TT}/E_T	0.2667	0.4
ν_{LT}	0.22	0.25

$h/L = 0.01$

Boundary conditions

At $x_1 = 0, L$
$w = \phi_2 = 0$

At $x_2 = 0, L$
$w = \phi_1 = 0$

Fig. 1. Panels used in the present study and sign convention for displacements and rotations.

cross-ply panel, and a 4×8 grid was used for the angle-ply panel. For both panels approximate analytic solutions, based on the combined application of the Bubnov–Galerkin and harmonic balance techniques were presented in [32]. Typical results are shown in Fig. 2 for the cross-ply panel and in Figs. 3 and 4 for the angle-ply panels, and are discussed subsequently.

In Fig. 2 comparisons are made between the nonlinear frequencies and kinetic energies, associated with the first symmetric/symmetric and first symmetric/antisymmetric modes, obtained by the reduced basis technique and those obtained by the perturbation technique. The frequencies obtained by the four-mode approximate analytical technique of [32] are also shown. The in-plane inertia, rotatory inertia and transverse shear deformation are neglected in [32]. However, the agreement between the frequencies reported in [32] and the corresponding ones obtained by the reduced basis technique is reasonable. This is particularly true for the first symmetric/symmetric mode. The rapid convergence of the solutions obtained by the

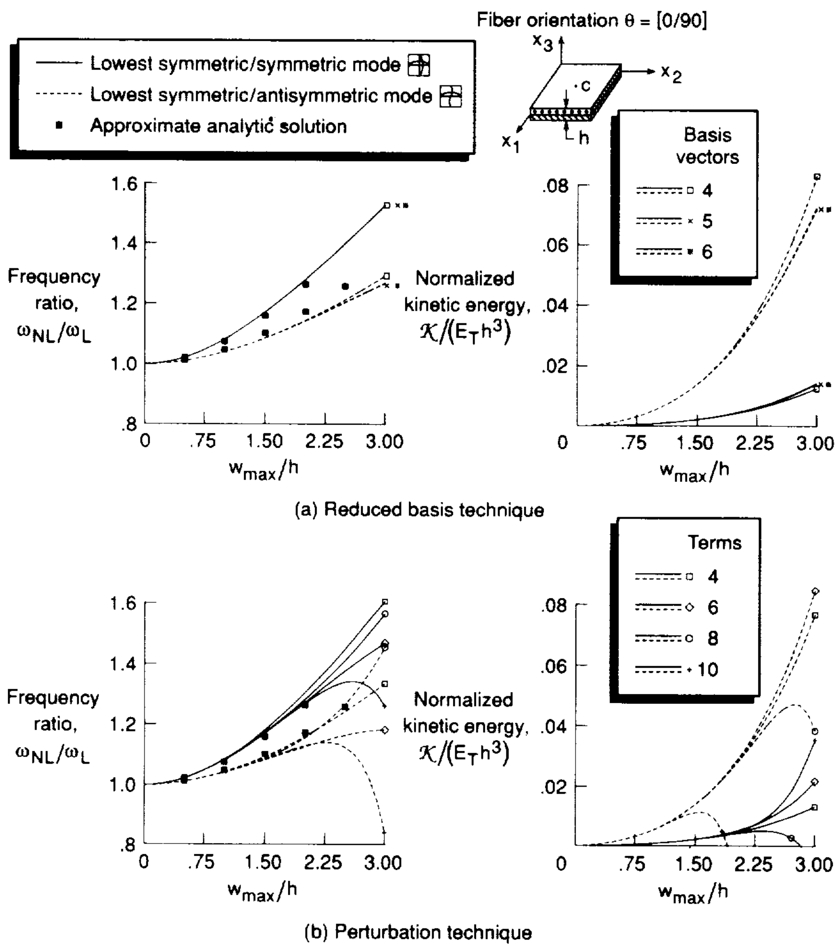


Fig. 2. Convergence of nonlinear frequencies and kinetic energies obtained by reduced basis and perturbation techniques. Two-layer cross-ply square plate made of material 1 (see Fig. 1).

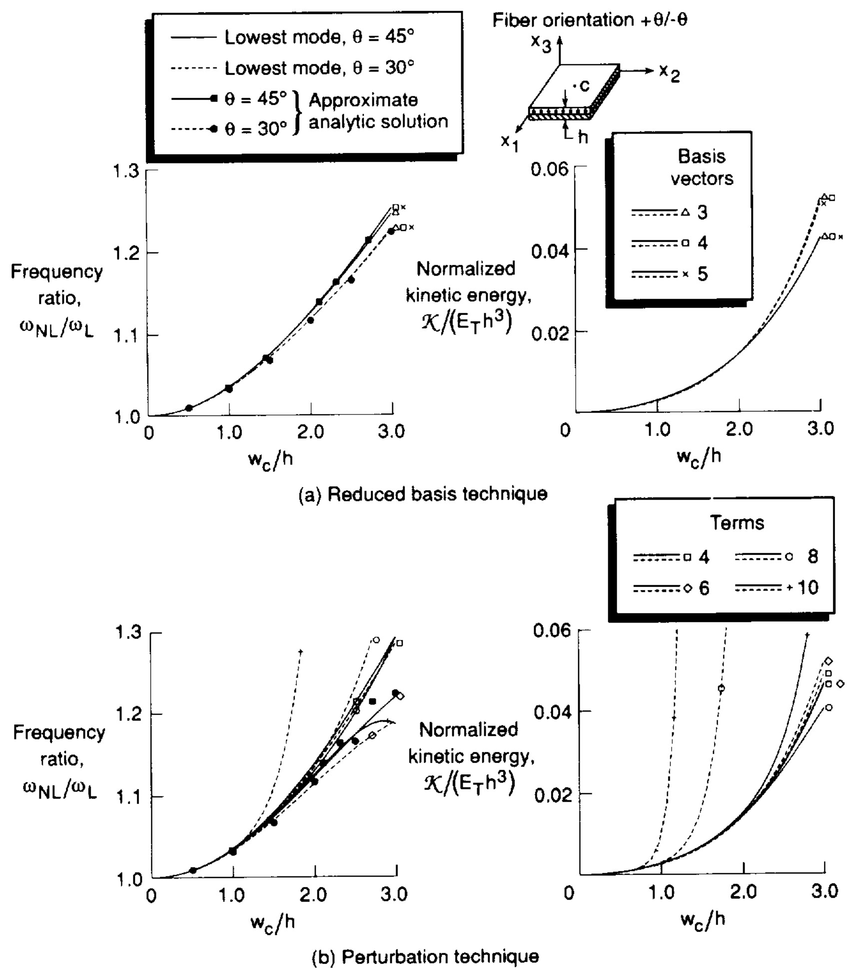
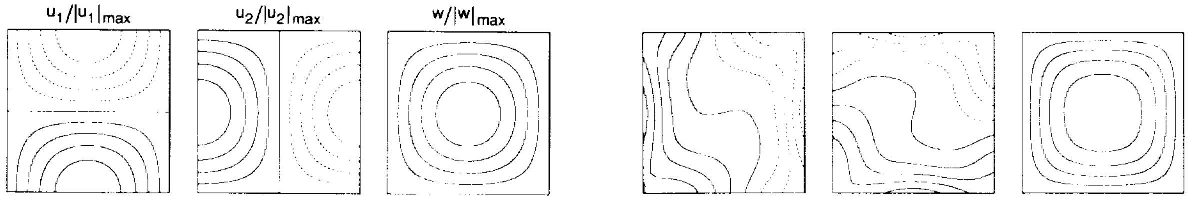


Fig. 3. Convergence of nonlinear frequencies and kinetic energies obtained by reduced basis and perturbation techniques. Two-layer angle-ply square plate made of material 2 (see Fig. 1).

reduced basis technique, for amplitudes up to three times the thickness of the plate, $w/h = 3.0$, is demonstrated. The perturbation solutions diverge for $w/h > 1.5$.

In Fig. 3 comparisons are made between the nonlinear frequencies and kinetic energies, associated with the fundamental vibration mode, obtained by the reduced basis and perturbation technique and those of [32], for angle-ply composite panels with $\theta = 30°$ and 45°. As for the cross-ply panel, the solutions obtained by the reduced-basis technique are in close agreement with those of [32], and converge rapidly for amplitudes up to three times the thickness of the plate. The perturbation solutions diverge for $w_c/h > 1.0$.

In Fig. 4 normalized contour plots are presented for the displacements u_1, u_2 and w associated with the linear fundamental vibration mode, and the nonlinear mode at $w_c/h = 3.0$. As expected, the effect of nonlinearity on u_1 and u_2 is much more pronounced than that on w.



(a) Fundamental vibration mode - linear
$(|u_1|_{max}/|w|_{max} = |u_2|_{max}/|w|_{max} = 7.46 \times 10^{-3})$

(b) Fundamental vibration mode - nonlinear, $w_c/h = 3.0$
$(|u_1|_{max}/|w|_{max} = |u_2|_{max}/|w|_{max} = 3.2 \times 10^{-2})$

Fig. 4. Normalized contour plots for the displacements associated with the fundamental linear and nonlinear modes, for two-layer angle-ply square plate made of material 2, $\theta = 45°$ (see Fig. 1). Spacing of contour lines is 0.2 and dashed lines refer to negative contours.

4. Potential of the reduced basis technique

The reduced basis technique has high potential for solution of nonlinear free vibration problems of composite panels with complex geometry. The numerical studies conducted clearly demonstrate the rapid convergence of the technique, well beyond the range of applicability of the perturbation technique. In addition, the technique can be extended to the evaluation of the sensitivity coefficients (derivatives of the nonlinear vibrational response with respect to material, lamination and geometric parameters). The extension is highlighted in this section.

The sensitivity coefficients are obtained by differentiating the governing finite element equations, eq. (1), with respect to d_l, where d_l denotes typical material, lamination or geometric parameter. The unknowns in the resulting linear equations,

$$\frac{\partial}{\partial d_l}\begin{Bmatrix}H\\X\end{Bmatrix},$$

are approximated by the following linear combination of basis vectors:

$$\frac{\partial}{\partial d_l}\begin{Bmatrix}H\\X\end{Bmatrix} = \sum_{i=1}^{r}\left[\left(\sum_{m=0}^{i}\begin{Bmatrix}H\\X\end{Bmatrix}^{(i,m)}\cos m\omega t\right)\bar{\psi}_i + \left(\sum_{m=0}^{i}\frac{\partial}{\partial d_l}\begin{Bmatrix}H\\X\end{Bmatrix}^{(i,m)}\cos m\omega t\right)\psi_i\right], \tag{7}$$

where $\{{H \atop X}\}^{(i,m)}$ are the basis vectors used in approximating the response;

$$\frac{\partial}{\partial d_l}\begin{Bmatrix}H\\X\end{Bmatrix}^{(i,m)}$$

are the sensitivity coefficients of the basis vectors; $\bar{\psi}_i$ and ψ_i are the amplitudes of the basis vectors for the sensitivity coefficients and the response, respectively. The sensitivity coefficients of the basis vectors,

$$\frac{\partial}{\partial d_l}\begin{Bmatrix}H\\X\end{Bmatrix}^{(i,m)},$$

are obtained by differentiating the recursive set of equations, eq. (5), with respect to d_l.

Equations (6) and (7) are used, in conjunction with the Bubnov–Galerkin technique and the method of harmonic balance, to approximate the full system in the sensitivity coefficients,

$$\frac{\partial}{\partial d_l}\begin{Bmatrix} H \\ X \end{Bmatrix},$$

by a reduced system of r linear algebraic equations in $\bar{\psi}_i$ $(i = 1, \ldots, r)$ and $\partial\omega^2/\partial d_l$. The additional equation needed to solve the system is obtained by differentiating the constraint condition used in evaluating ψ_i and ω (i.e., in generating the nonlinear vibrational response) with respect to d_l.

5. Concluding remarks

A reduced basis technique and a computational procedure are presented for the nonlinear vibration analysis of composite panels. The technique is based on the successive use of regular perturbation method and direct variational procedure. The computational procedure can be conveniently divided into two distinct steps. In the first step, perturbation vectors are generated using the Linstedt–Poincaré perturbation method. In the second step, the perturbation vectors are used as basis vectors. The amplitudes of these vectors and the nonlinear frequency are computed using a direct variational procedure in conjunction with the method of harmonic balance. The analytical formulation is based on a form of the geometrically nonlinear shallow shell theory with the effects of transverse shear deformation, in-plane inertia, rotatory inertia and anisotropic material behavior included. The panel is discretized by using mixed finite element models with the fundamental unknowns consisting of both the nodal displacements and the stress-resultant parameters of the panel. The effectiveness of the reduced basis technique is demonstrated by means of numerical examples of cross-ply and angle-ply composite panels. The frequencies obtained by the reduced basis technique were shown to be close to the corresponding ones obtained by the approximate analytic technique well beyond the range of applicability of the perturbation technique. The potential of the reduced basis technique for evaluating the sensitivity coefficients of the nonlinear vibrational response is discussed.

On the basis of the present study the following two observations can be made:

(1) The reduced basis technique can be thought of as either of the following: (a) generalized perturbation method in which (i) perturbation expansions of the nodal displacements and stress resultants contain free parameters rather than fixed coefficients and (ii) the perturbation parameters need not be small; (b) an extended direct variational technique with the coordinate functions generated by using a perturbation technique rather than chosen a priori.

(2) The successive application of the perturbation method and the direct variational technique, which forms the basis of the proposed computational procedure, results in (a) enhancing the effectiveness of the direct variational technique by removing (or reducing) the arbitrariness in the selection of the approximation vectors, and (b) extending the range of applicability of the regular perturbation method by removing the restriction of a small perturbation parameter.

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Appendix A. Evaluation of perturbation vectors

For individual elements, the nonlinear terms G and $\bar{G}$ in (1) are expressed in the following form (see [38]):

$$G_I = \tfrac{1}{2} N_{IJ'K'} X_{J'} X_{K'} , \tag{A.1}$$

$$\bar{G}_{I'} = N_{KI'J'} H_K X_{J'} . \tag{A.2}$$

In (A.1) and (A.2) $N_{IJ'K'}$ represent nonlinear contributions which consist of integrals, over the element domain, of products of shape functions and their spatial derivatives (see [38]). The uppercase Latin subscripts I, J, K range from 1 to the total number of stress-resultant parameters in the element; I', J', K' range from 1 to the total number of nodal displacements in the element; and a repeated index denotes summation over its full range. The perturbation vectors are obtained by solving a recursive set of equations which, for individual elements, can be written in the following compact form:

$$\begin{bmatrix} -F_{IJ} & S_{IJ'} \\ (S_{IJ'})^{\mathrm{t}} & -m^2\omega^2 M_{I'J'} \end{bmatrix} \begin{Bmatrix} H_J \\ X_{J'} \end{Bmatrix}^{(i,m)} = -\begin{Bmatrix} R_I \\ Q_{I'} \end{Bmatrix}^{(i,m)} . \tag{A.3}$$

The explicit forms of the components of the right-hand sides, R_I and $Q_{I'}$, for each pair of (i, m) (see (2)–(5)), are given in Table 2.

In Table 2 a repeated uppercase Latin index denotes summation over its full range and the coefficient c_1 is given by

$$c_1 = -\sum_{\text{elements}} (H_I^{(1,1)} R_I^{(3,1)} + X_{I'}^{(1,1)} \bar{Q}_{I'}^{(3,1)}) / (M_{I'J'} X_{I'}^{(1,1)} X_{J'}^{(1,1)}) , \tag{A.4}$$

with

$$\bar{Q}_{I'}^{(3,1)} = N_{KI'J'} (2H_K^{(1,1)} X_{J'}^{(2,0)} + 2H_K^{(2,0)} X_{J'}^{(1,1)} + H_K^{(1,1)} X_{J'}^{(2,2)} + H_K^{(2,2)} X_{J'}^{(1,1)}) , \tag{A.5}$$

$$Q_{I'}^{(3,1)} = \bar{Q}_{I'}^{(3,1)} + c_1 M_{I'J'} X_{J'}^{(1,1)} . \tag{A.6}$$

Note that when $m = 1$, the left-hand side matrix is singular. In this case, the vector

$$\begin{Bmatrix} H_J \\ X_{J'} \end{Bmatrix}^{(i,m)}$$

can be written as the sum of a particular solution plus a multiple of the eigenvector. For example, for $m = 1$, $i = 3$, the vector

$$\begin{Bmatrix} H_J \\ X_{J'} \end{Bmatrix}^{(3,1)}$$

is given by

$$\begin{Bmatrix} H_J \\ X_{J'} \end{Bmatrix}^{(3,1)} = \begin{Bmatrix} \bar{H}_J \\ \bar{X}_{J'} \end{Bmatrix}^{(3,1)} + \alpha \begin{Bmatrix} H_J \\ X_{J'} \end{Bmatrix}^{(1,1)} , \tag{A.7}$$

where the first term on the right-hand side of (A.7) refers to the particular solution with one component fixed, and α is given by

$$\alpha = - \sum_{\text{elements}} M_{I'J'} \bar{X}_{I'}^{(3,1)} X_{J'}^{(1,1)} / (M_{I'J'} X_{I'}^{(1,1)} X_{J'}^{(1,1)}) . \tag{A.8}$$

Appendix B. Form of the nonlinear equations in ψ_i and ω

The nonlinear algebraic equations in the amplitudes of the coordinate functions ψ_i, and the frequency ω, can be written in the following compact form:

$$\bar{K}_{ij}\psi_j + \bar{F}_{ijk}\psi_j\psi_k - \omega^2 \bar{M}_{ij}\psi_j = 0 , \tag{B.1}$$

where the range of i, j, k is $1, \ldots, r$; and a repeated index in the same term denotes summation over its full range. The arrays $\bar{K}_{ij}$, $\bar{F}_{ijk}$ and $\bar{M}_{ij}$ are obtained by using (6) and (1); applying the Bubnov–Galerkin method and the method of harmonic balance; and performing the temporal integration.

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